Lecture 3:

Recap:

① Image transformation : $\mathcal{O} : \mathcal{I} \to \mathcal{I}$

Space of images

Linear if :

$$\mathcal{O}(af + g) = a\,\mathcal{O}(f) + \mathcal{O}(g)$$

real number

images

② If $g = \mathcal{O}(f)$ (f and g are images)

then: For any $1 \le \alpha, \beta \le N$,

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, \overbrace{h(x, \alpha, y, \beta)}^{\text{point spread function}}$$

where

$$h(x, \alpha, y, \beta) = \left[\mathcal{O}(P_{xy})\right]_{\alpha,\beta} ; \qquad P_{xy} = \begin{array}{cc} \begin{matrix} & & y\text{-th} & & \\ & & \downarrow & & \end{matrix} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \leftarrow x^{th} \end{array}$$

③ Shift-invariant if:

$$h(x,\alpha,y,\beta) = \tilde{h}(\alpha-x,\beta-y)$$ ($\tilde{h}(\cdot,\cdot)$ depends on two variables)

④ Separable if:

$$h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$$ (Both $h_c(\cdot,\cdot)$ and $h_r(\cdot,\cdot)$ depends on two variables)

⑤ If $g = \mathcal{O}(f)$ and if $\mathcal{O}$ is shift invariant (assuming that $h(x,\alpha,y,\beta) = \tilde{h}(\alpha-x,\beta-y)$)

then: $g = f * \tilde{h}$ ← convolution

$$\left(f*\tilde{h}(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h(\alpha-x,\beta-y) \right)$$

⑥ If $g = \mathcal{O}(f)$ and if $\mathcal{O}$ is separable, (assuming that $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$)

then: $g = h_c^T f h_r$ (Regarding $h_c(\cdot,\cdot)$, $h_r(\cdot,\cdot)$, f as matrices)

Separable means two matrix multiplications)

⑦ If $g = \mathcal{O}(f)$ and if we write:

$$\vec{f} = \begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{N1} \\ f_{12} \\ f_{22} \\ \vdots \\ f_{N2} \\ \vdots \\ f_{1N} \\ \vdots \\ f_{NN} \end{pmatrix} \quad \text{and} \quad \vec{g} = \begin{pmatrix} g_{11} \\ \vdots \\ g_{N1} \\ g_{12} \\ \vdots \\ g_{N2} \\ \vdots \\ g_{1N} \\ \vdots \\ g_{NN} \end{pmatrix}.$$

Then: $\vec{g} = H \vec{f}$

H is a $N^2 \times N^2$ big matrix!!

By careful examination, we see that:

$$
\begin{array}{cc}
 & y \longrightarrow \\
\beta \Big\downarrow & H = \left(\begin{array}{cccc}
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1\end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1\end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=1\end{pmatrix}\end{array}\right) \\
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2\end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2\end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=2\end{pmatrix}\end{array}\right) \\
\vdots & \vdots & & \vdots \\
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=N\end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=N\end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=N\end{pmatrix}\end{array}\right)
\end{array}\right)
\end{array}
$$

Meaning of

col of small block ↓, row of small block ↓, col of block matrix ↙, row of block matrix ↙

$$h(x, \alpha, y, \beta)$$

$$
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j\end{pmatrix}\end{array}\right) : \begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix} \in M_{N\times N}
$$

Definition: H is called the transformation matrix of $\mathcal{O}$.

Example 1.2 Consider an image transformation on a 2×2 image. Suppose the matrix representation of the image transformation is given by:

$$H = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 \\ 3 & 0 & 4 & 0 \\ 6 & 3 & 8 & 4 \end{pmatrix}.$$

Prove that the image transformation is separable. Find g_1 and g_2 such that:

$$h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta).$$

Solution: H for a 2×2 image :

$$\begin{pmatrix} \alpha\downarrow\overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}} & \alpha\downarrow\overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}} \\ \alpha\downarrow\overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}} & \alpha\downarrow\overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}} \end{pmatrix} \in M_{4\times 4}$$

H is separable $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g_1(x,\alpha)\, g_2(y,\beta)$.

Easy to check:

if H is separable: $H = \begin{pmatrix} g_2(1,1) G_1 & g_2(2,1) G_1 \\ g_2(1,2) G_1 & g_2(2,2) G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_1(2,2) \end{pmatrix}$

$$\begin{pmatrix} \begin{pmatrix} h(1,1,1,1) & h(2,1,1,1) \\ h(1,2,1,1) & h(2,2,1,1) \end{pmatrix} & \begin{pmatrix} h(1,1,2,1) & h(2,1,2,1) \\ h(1,2,2,1) & h(2,2,2,1) \end{pmatrix} \\ \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} & \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} g_1(1,1)g_2(1,1) & g_1(2,1)g_2(1,1) \\ g_1(1,2)g_2(1,1) & g_1(2,2)g_2(1,1) \end{pmatrix} = g_2(1,1) \underbrace{\begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_2(1,1) \end{pmatrix}}_{G_1}$$

$$g_2(2,1)\, G_1$$

In our case, $H = \begin{pmatrix} 2G_1 & 1G_1 \\ 3G_1 & 4G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\therefore$

$g_1(1,1) = 1$	$g_2(1,1) = 2$
$g_1(2,1) = 0$	$g_2(2,1) = 1$
$g_1(1,2) = 2$	$g_2(1,2) = 3$
$g_1(2,2) = 1$ //	$g_2(2,2) = 4$ //

Properties of shift-invariant/separable image transformation

Definition: (Circulant matrix)

A circulant matrix $V := \mathrm{circ}(\vec{v})$ associated to a vector $\vec{v} = (v_0, v_1, \ldots, v_{n-1})^T \in \mathbb{C}^n$ is a $n \times n$ matrix whose columns are given by iterations of shift operator T acting on $\vec{v}$. Here, $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_0 \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix}.$$

$\therefore$ k^{th} column is given by $T^{k-1}(\vec{v})$ $(k = 1, 2, \ldots, n)$

$$\therefore \quad V = \begin{pmatrix} v_0 & v_{n-1} & v_{n-2} & \cdots & v_1 \\ v_1 & v_0 & v_{n-1} & \cdots & v_2 \\ \vdots & \vdots & v_0 & & \vdots \\ & & \vdots & & v_{n-1} \\ v_{n-1} & v_{n-2} & v_{n-3} & \cdots & v_0 \end{pmatrix}$$

Definition:(Block circulant)

V is block-circulant $\Leftrightarrow$ $V = \begin{pmatrix} H_0 & H_{n-1} & \cdots & H_1 \\ H_1 & H_0 & & H_2 \\ \vdots & \vdots & \cdots & \vdots \\ & & & H_{n-1} \\ H_{n-1} & H_{n-2} & \cdots & H_0 \end{pmatrix} \ni$

each H_i is a circulant matrix.

Theorem: If H = transf. matrix of shift-invariant operator,

then $H = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ A_{21} & \cdots & A_{2N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$ where each A_{ij} is a circulant matrix.

(Assuming $h(x, \alpha, y, \beta) = g(\alpha - x, \beta - y)$ and g is periodic in 1st and 2nd argument)

Proof:

Consider $A_{ij} = \begin{array}{cc} & x \rightarrow \\ \begin{array}{c}\alpha\\ \downarrow\end{array} & \left(\begin{array}{c} y = j \\ \beta = i \end{array} \right) \end{array}$

$$\therefore A_{ij} = \begin{pmatrix} h(1,1,j,i) & h(2,1,j,i) & \cdots & h(N,1,j,i) \\ h(1,2,j,i) & h(2,2,j,i) & \cdots & h(N,2,j,i) \\ \vdots & \vdots & & \vdots \\ h(1,N,j,i) & h(2,N,j,i) & \cdots & h(N,N,j,i) \end{pmatrix}$$

Shift-invariant $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g(\alpha - x, \beta - y)$ for some g.

$$\therefore A_{ij} = \begin{pmatrix} g(0, i-j) & g(\overset{N-1}{\cancel{-1}}, i-j) & \cdots & g(\overset{1}{\cancel{1-N}}, i-j) \\ g(1, i-j) & g(0, i-j) & \cdots & g(\overset{2}{\cancel{2-N}}, i-j) \\ \vdots & \vdots & & \vdots \\ g(N-1, i-j) & g(N-2, i-j) & \cdots & g(0, i-j) \end{pmatrix} \leftarrow \text{Circulant}$$

(Assume periodic property)

Image decomposition

If $f = A g B$ (f and g are images; A and B are matrices) (All f, g, A, B are ($N \times N$ matrices))

(Separable image transformation)

then: $$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{a}_i \, \vec{b}_j^{T}}_{\text{elementary images}}$$

where $g_{ij} = i^{th}$ row, j^{th} col of g

$$A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \end{pmatrix} ; \quad B = \begin{pmatrix} \vec{b}_1^{T} \\ \vec{b}_2^{T} \\ \vdots \\ \vec{b}_N^{T} \end{pmatrix}$$

Example: Let $$f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + 3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + 5 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

Then: $$f = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix}$$

Remark: Separable image transformation allows us to write an image as a linear combination of elementary images!!

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m\times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\text{rank}(AB) = \text{rank}(B)$ if A is invertible

$\text{rank}(AB) = \text{rank}(A)$ if B is invertible.

Suppose $g = U\Sigma V^T$. Since U and V are invertible, $\text{rank}(g) = \text{rank}(\Sigma)$

$= \#$ of non-zero singular values

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Later!

Remark: Suppose $A = U \Sigma V^T$ $(\in M_{m \times n})$ (SVD decomposition) where
$$\Sigma = \begin{array}{c} n \\ \left(\begin{array}{cccc} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ \hline & & 0 & \end{array}\right) \end{array} \; m$$

1. $$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T U^T U \Sigma V^T$$
$$= V \Sigma^T \Sigma V^T$$

$$\therefore (A^T A) V = V \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_r^2 \end{pmatrix}$$

If $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$, then: $(A^T A) \vec{v}_j = \sigma_j^2 \vec{v}_j$

($\vec{v}_j$ is the eigenvector of $A^T A$ with eigenvalue σ_j^2)

Also, $\|\vec{v}_j\| = 1$ as $V^T V =$ Identity.

2. Similarly, if $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_m \\ | & | & & | \end{pmatrix}$, then $\vec{u}_i$ = eigenvector of AA^T with eigenvalue σ_i^2

Also, $\|\vec{u}_i\| = 1$

3. Let $\vec{u}_i = (A\vec{v}_i)/\sigma_i$. Then: $\|\vec{u}_i\| = \vec{u}_i^T \vec{u}_i = \frac{\vec{v}_i^T A^T}{\sigma_i} \frac{A\vec{v}_i}{\sigma_i} = \frac{\vec{v}_i^T \sigma_i^2 \vec{v}_i}{\sigma_i^2} = \frac{\sigma_i^2}{\sigma_i^2} = 1$

Also, $AA^T\vec{u}_i = AA^T(A\vec{v}_i/\sigma_i) = A\sigma_i^2\vec{v}_i/\sigma_i$

$= \sigma_i^2 A\vec{v}_i/\sigma_i = \sigma_i \vec{u}_i$

$\therefore \vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$

How to compute SVD

Let $A \in M_{m\times n}$ $(m \geq n)$ $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ $\rightarrow V$

of symmetric $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \underbrace{\sqrt{\lambda_1}}_{=\sigma_1} & & & \\ & \underbrace{\sqrt{\lambda_2}}_{=\sigma_2} & & \\ & & \ddots & \\ & & & \underbrace{\sqrt{\lambda_n}}_{=\sigma_n} \\ \hline & & 0 & \end{pmatrix} \in M_{m\times n}$$

Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\dots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the o.n. basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$. $\rightarrow U$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} .$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix} . \quad \text{(Step 1)}$$

Now, eig(A^TA) are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix} .$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \ \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

Since

$$\sigma_1 \vec{u}_1 = A\vec{v}_1 ,$$

we have

$$\vec{u}_1 = \underbrace{\frac{1}{\sqrt{17}}}_{\sigma_1} \frac{1}{\sqrt{2}} \overbrace{\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}}^{A} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v}_1} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} .$$

$$\frac{A\vec{v}_2}{\sigma_2}$$

Similarly, we have

$$\vec{u}_2 \overset{\parallel}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \\ \frac{4}{\sqrt{34}} & 0 & \mathbf{u}_3 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \end{pmatrix}$$

for some vector $\mathbf{u}_3$ orthonormal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. One possibility is

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} .$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} .$$

Image decomposition by SVD:

- Note that $g = U\Lambda^{1/2}V^T = \sum_{i=1}^{r} \sigma_i\, U \begin{array}{c} \scriptstyle i\text{th} \\ \downarrow \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \end{array} \leftarrow i\text{th} \; V^T = \sum_{i=1}^{r} \sigma_i \underbrace{\vec{u}_i \vec{v}_i^T}_{\text{elementary images}}$

  $\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

- For $N \times N$ image, the required storage is:

$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r\text{ terms}} = (2N+1)r$$

- Sometimes, we may only keep the first few terms in the decomposition to further save the storage:

$$\sum_{i=1}^{k} \sigma_i \underbrace{\vec{u}_i \vec{v}_i^T}_{\text{elementary images}} \quad \text{where } k < r$$